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On the motion of suspended particles in stationary homogeneous turbulence

By L. M. PISMEN AND A. NIR
Department of Chemical Engineering, Israel Institute of Technology, Haifa

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The closed equations for the velocity correlation tensor and for the mean-squared displacement of a particle suspended in a stationary homogeneous turbulent flow, with an arbitrary linear law of fluid–particle interaction, are obtained using two assumptions suggested previously for the problem of turbulent self-diffusion: the 'independence approximation' and the Gaussian property of the functional distribution of particle velocities. The numerical solution of the derived equations is given for an isotropic system with a model turbulence spectrum. The following characteristics of the particle motion are obtained: (*a*) the mean kinetic energy, (*b*) diffusivity, (*c*) rate of energy dissipation, (*d*) velocity correlation function, and (*e*) the correlation function of the relative fluid–particle velocity. The impact of various spectral modes on the characteristics of the particle motion is discussed.

1. Introduction

A solid particle suspended in turbulent flow fluctuates by responding to fluctuations in the velocity of the surrounding fluid. This response is governed by the equation of motion

$$dw_i/dt = F_i(\mathbf{w}(t), \mathbf{u}(\mathbf{r}(t), t)), \tag{1.1}$$

which includes the force per unit particle mass F_i expressed as a functional of the fluid and particle velocities, u_i and w_i, with r_i denoting the particle position. A solution of the problem of the particle motion is implied by finding statistical characteristics of the random function $w_i(t)$ given the statistical description of the turbulent velocity field $u_i(\mathbf{r}, t)$ and the interaction law $F_i(\mathbf{w}, \mathbf{u})$.

In general the functional F_i is a complicated nonlinear expression. For the particular case of local linear interaction with a negligible 'history term', (1.1) reduces to the quite innocuous form

$$dw_i/dt = \gamma(u_i - w_i). \tag{1.2}$$

Yet even this relation, as was early realized (Lumley 1957; Friedlander 1957; Soo 1967), is haunted by intrinsic nonlinearity. The fluid velocity $u_i(\mathbf{r}(t))$ appearing in (1.1) and (1.2) can be viewed as a profile of the random field $u_i(\mathbf{r}, t)$ along the unknown particle trajectory $r_i(t)$. While affecting the trajectory, the function $u_i(\mathbf{r}(t))$ is, in its turn, determined by the random particle movements, so that (1.2), contrary to its appearance, is actually an extremely complicated nonlinear functional equation.

This basic difficulty was avoided (Tchen 1947) by assuming that a particle is permanently confined to the same fluid element. Such an assumption immediately allows the use of Lagrangian correlations of flow velocity when solving (1.2) or its more sophisticated versions. An obvious corollary, independent of the form of the dynamic equations, is the coincidence of the long-time particle diffusivity with the diffusivity of the fluid element. This conclusion holds in some cases, but it is exactly the difference in the motion of fluid elements and foreign particles which is most interesting to evaluate.

While Tchen's (1947) results are still widely used and cited (Levins & Glastonbury 1972; Kuboi, Komasawa & Otake 1974) attempts have been made to improve the theory using the intuitive concept of random encounters between a moving particle and fluid elements (Soo 1967) and the notion of crossing trajectories (Yudine 1959; Csanady 1963; Meek & Jones 1973). These works have elucidated two basic mechanisms for the discrepancy between the diffusivities of fluid elements and foreign particles: a trend to increasing particle diffusivity due to inertial effects and a counteracting trend, usually stronger, to decreasing diffusivity due to the passing of a particle from one domain of strongly correlated fluid to another.

Despite the undisputable usefulness of semi-quantitative theories, a treatment of the problem of particle motion does not appear fully satisfactory, unless it starts from 'first principles', i.e. it is based on a direct solution of the stochastic dynamic equations (1.1) or (1.2). On the other hand, it may be argued that even an exact solution of these equations, were it ever found, would be practically useless. The most important characteristics of particle motion, notably, the velocity correlation function determining the particle diffusivity, should depend not only on two-point correlation functions of turbulent flow, either Lagrangian or Eulerian, but on a whole hierarchy of correlations. Thus any reasonable set of data on turbulent flow, let alone data reported in the most comprehensive experimental studies of particle diffusion (Snyder & Lumley 1971; Goldschmidt & Householder 1969; Lilly 1973), would not suffice for implementation of the hypothetical exact theory, and one would have to wait for the arrival of a complete statistical theory of turbulence, no less hypothetical, to secure all relevant input data.

A feasible middle way of approaching the problem consists of solving stochastic dynamic equations while assuming certain statistical hypotheses for the distribution of functional probabilities, akin to closure hypotheses in the theory of turbulence. This would allow solutions to be found in closed form excluding higher-order terms in the hierarchy of correlations. Such an approach, initiated by Saffman (1963), was recently applied by Lundgren & Pointin (1976) to the problem of turbulent self-diffusion and it involves two major assumptions: (1) the characteristic function of the velocity field is Gaussian, i.e., it depends only on the second-order correlation $\langle u_i(0,0)\,u_j(\mathbf{r},t)\rangle$; (2) Corrsin's (1959) conjecture or an equivalent 'independence approximation' can be implemented, expressing the Lagrangian correlation tensor as the average of the Eulerian correlation function taken with respect to the uncertain position of the particle.

Calculations based on the above assumptions compare well with Kraichnan's (1970) exact solutions for model turbulent spectra. The theoretical justification of these assumptions was given by Weinstock (1976). In this work we suggest that this approach, which has proved satisfactory for the characterization of the motion of fluid elements, applies to the problem of particle motion as well. We start in §2 by transforming the dynamic equations in order to express the particle–velocity correlation tensor in terms of the fluid–velocity correlation tensor at the points lying on a particle trajectory. In § 3, using the two aforementioned assumptions as applied to the functional distribution of particle trajectories, we derive the closed equation of the mean-squared particle displacement valid for the stationary motion of a particle in a stationary homogeneous turbulence with a linear law of fluid–particle interaction. An explicit numerical solution of the derived equation is performed in § 4 for an isotropic system governed by the dynamic equation (1.2), using Kraichnan's (1970) model spectra of turbulence. The following characteristics of particle motion are also calculated: (*a*) the mean kinetic energy, (*b*) diffusivity, (*c*) velocity correlation function, (*d*) mean energy dissipation, and (*e*) the correlation function of the relative velocity of particle and surrounding fluid. These results are then discussed in the last section. Effects of anisotropy induced by a deterministic part of the particle velocity will be discussed in a later communication.

2. Transformation of the dynamic equations

Consider the dynamic equation (1.1) with a linear functional $F_i(\mathbf{u}, \mathbf{w})$. Introducing an appropriate Green's function $K_{ij}(t-t')$, (1.1) can be rewritten in the integral form

$$w_i(t) - w_i(0) = \int_0^t K_{ij}(t-t')\, u_j(\mathbf{r}(t'), t')\, dt'. \tag{2.1}$$

Summation over repeating indices is assumed here and elsewhere. It is convenient to use the co-ordinate framework moving with the mean flow velocity. The dependence of the Green's function K_{ij} only on the difference $t-t'$ is assured by the spatial and temporal invariance of the dynamic law of particle–fluid interaction. It is also useful to separate the deterministic and random parts of the particle velocity $w_i(t)$:

$$w_i(t) = v_{0i} + v_i(t), \quad \langle v_i(t) \rangle = 0, \tag{2.2}$$

with the bracketed term denoting the ensemble average. For a stationary particle motion the deterministic part v_{0i} should be constant, while the random part $v_i(t)$ is independent of the initial conditions. Separating also the random part of the particle displacement,

$$\rho_i(t) = r_i(t) - v_{0i}\, t, \quad \langle \rho_i(t) \rangle = 0, \tag{2.3}$$

we reduce (2.1) to the form

$$v_i(t) = \int_0^t K_{ij}(t-t')\, u_j(\mathbf{v}_0 t' + \boldsymbol{\rho}(t'), t')\, dt'. \tag{2.4}$$

A particular form of the Green's function, corresponding to the dynamic equation (1.2), is

$$K_{ij}(t) = \gamma e^{-\gamma t} \delta_{ij}. \tag{2.5}$$

Although (1.2) is formally equivalent to an equation of motion for a Stokesian particle, Oseen's inertial effects may be accounted for by merely modifying the particle time constant γ^{-1}. Boussinesq's effect may be included by incorporating a complex time constant into the time Fourier transforms of (2.4) and (2.5). Moreover, a nonlinear interaction law may be reduced to the form (2.5) with a tensorial γ if it can be linearized with respect to the random part of the relative velocity $w_i - u_i$ (Lumley 1976). Thus, one can expect the Green's function (2.5), with an appropriate time constant, to be valid as long as the Reynolds number based on a characteristic pulsation velocity does not exceed unity. More complicated kernels will appear in (2.1) if the history term is retained or when non-local effects due to the finite size of a particle are included in the dynamic equation.

Using (2.4), the particle velocity correlation tensor can be written as

$$H_{ij}(s) = \lim_{t\to\infty} \langle v_i(t)\, v_j(t+s)\rangle = \lim_{t\to\infty} \left\langle \int_0^t K_{ik}(t-t')\, u_k(t')\, dt' \int_0^{t+s} K_{jl}(t+s-t'')\, u_l(t'')\, dt'' \right\rangle$$

$$= \lim_{t\to\infty} \int_0^t K_{ik}(\tau)\, d\tau \int_{-(t-\tau)}^{\tau+s} K_{jl}(\tau+s-s')\, G_{kl}(s')\, ds', \qquad (2.6)$$

where $s' = t'-t''$ and

$$G_{kl}(t) = \langle u_k(0,0)\, u_l(\mathbf{v}_0 t + \boldsymbol{\rho}(t), t)\rangle \qquad (2.7)$$

is the fluid velocity correlation tensor taken at points lying on a particle trajectory.

Since we are interested in stationary characteristics of the particle motion, the $t\to\infty$ limit should be considered. Changing the order of integration in the double integral in (2.6) yields

$$H_{ij}(s) = \int_{-\infty}^{\infty} L_{ikjl}(s-s')\, G_{kl}(s')\, ds', \qquad (2.8)$$

where

$$L_{ikjl}(x) = \int_{a(x)}^{\infty} K_{ik}(\tau)\, K_{jl}(\tau-x)\, d\tau, \qquad (2.9)$$

with $a(x) = 0$ at $x < 0$ and $a(x) = x$ at $x > 0$.

For the particular Green's function in (2.5) the particle velocity correlation tensor is given by the equation

$$H_{ij}(s) = \tfrac{1}{2}\gamma \int_{-\infty}^{\infty} e^{-\gamma|s-s'|} G_{ij}(s')\, ds'. \qquad (2.10)$$

At $\gamma\to\infty$, when the particle exactly follows all flow pulsations, (2.10) gives, as expected, $H_{ij}(s) \equiv G_{ij}(s)$.

Equations (2.8) and (2.10) connect the particle velocity correlation tensor with the fluid velocity correlation tensor *at points lying on a particle trajectory*. These equations are exact, inasmuch as the dynamic equations involved in the transformation are valid, but still useless as long as the tensor $G_{ij}(s)$ is not reduced to measurable correlations of fluid velocity. In the following section the closed equation will be deduced using a procedure equivalent to that suggested by Lundgren & Pointin (1976).

3. The equation for the mean-squared displacement

Let the Fourier transform of the fluid velocity field be

$$u_i(\mathbf{r}, t) = \int_{-\infty}^{\infty} \tilde{u}_i(\mathbf{k}, t)\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^3\mathbf{k}. \qquad (3.1)$$

Then (2.7) may be written as

$$G_{ij}(t) = \iint_{-\infty}^{\infty} \exp\left[-i\mathbf{k}'.\mathbf{v}_0 t\right] \langle \tilde{u}_i(\mathbf{k}, 0)\, \tilde{u}_j(\mathbf{k}', t) \exp\left[-i\mathbf{k}'.\boldsymbol{\rho}(t)\right]\rangle\, d^3\mathbf{k}\, d^3\mathbf{k}'. \qquad (3.2)$$

The function $\exp\left[-i\mathbf{k}'.\boldsymbol{\rho}(t)\right]$ is affected by the *whole* spectrum of the fluid velocity field acting on the particle during the time interval t. If the dependence on any *particular* components, with wavenumbers $\mathbf{k}$ and $\mathbf{k}'$, is assumed negligible then

$$\langle \tilde{u}_i(\mathbf{k}, 0)\, \tilde{u}_j(\mathbf{k}', t) \exp\left[-i\mathbf{k}'.\boldsymbol{\rho}(t)\right]\rangle \cong \langle \tilde{u}_i(\mathbf{k}, 0)\, \tilde{u}_j(\mathbf{k}', t)\rangle \langle \exp\left[-i\mathbf{k}'.\boldsymbol{\rho}(t)\right]\rangle.$$

Equation (3.2) reduces to

$$G_{ij}(t) = \int_{-\infty}^{\infty} \exp[-i\mathbf{k}.\mathbf{v}_0 t]\, \Phi_{ij}(\mathbf{k},t)\, \langle \exp[-i\mathbf{k}.\boldsymbol{\rho}(t)]\rangle\, d^3\mathbf{k}, \tag{3.3}$$

where, with δ denoting the Dirac delta function,

$$\Phi_{ij}(\mathbf{k},t)\,\delta(\mathbf{k}+\mathbf{k}') = \langle \tilde{u}_i(\mathbf{k},0)\, \tilde{u}_j(\mathbf{k}',t)\rangle \tag{3.4}$$

is the spectral density of the fluid velocity field, or Fourier transform of the Eulerian correlation tensor

$$G^E_{ij}(\mathbf{r},t) = \int_{-\infty}^{\infty} \Phi_{ij}(\mathbf{k},t)\, e^{-i\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{k}. \tag{3.5}$$

The 'independence approximation' involved in deriving (3.3) from (3.2) was shown by Lundgren & Pointin (1976) and Weinstock (1976) to be equivalent to Corrsin's conjecture.

The average $\langle \exp[-i\mathbf{k}.\boldsymbol{\rho}(t)]\rangle$ is calculated using the assumption that the random function $v_i(t)$ is joint normal. The characteristic functional of a joint normal distribution is

$$\begin{aligned} Q(\mathbf{q}(t)) \equiv \left\langle \exp\left[-i\int_{-\infty}^{\infty} \mathbf{v}(t).\mathbf{q}(t)\, dt\right]\right\rangle \\ = \exp\left[-\tfrac{1}{2}\int_{-\infty}^{\infty} \mathbf{q}(t_1)\, \langle \mathbf{v}(t_1)\, \mathbf{v}(t_2)\rangle\, \mathbf{q}(t_2)\, dt_1\, dt_2\right]. \end{aligned} \tag{3.6}$$

Substituting

$$\mathbf{q}(t_1) = \mathbf{k} \quad \text{for} \quad 0 < t_1 < t \quad \text{and} \quad \mathbf{q}(t_1) = 0 \quad \text{otherwise} \tag{3.7}$$

in (3.6), we find

$$\begin{aligned} \langle \exp[-i\mathbf{k}.\boldsymbol{\rho}(t)]\rangle = \left\langle \exp\left[-i\mathbf{k}.\int_0^t \mathbf{v}(t)\, dt\right]\right\rangle = \exp\Bigg[-\tfrac{1}{2}\int_0^t dt_1 \int_0^t dt_2 \\ \times \mathbf{kk} : \langle \mathbf{v}(t_1)\, \mathbf{v}(t_2)\rangle\Bigg] = \exp\left[-\int_0^t (t-s)\, \mathbf{kk}:\mathbf{H}(s)\, ds\right]. \end{aligned} \tag{3.8}$$

The last transformation implies the stationarity of the particle velocity distribution and the symmetry property of the correlation tensor $H_{ij}(s) = H_{ji}(-s)$. With $G_{ij}(s)$ defined by (3.3) and (3.8), we find that (2.8) acquires the closed form

$$H_{ij}(t) = \int_{-\infty}^{\infty} L_{ikjl}(t-t')\, dt' \int_{-\infty}^{\infty} \exp[-i\mathbf{k}.\mathbf{v}_0 t']\, \Phi_{kl}(\mathbf{k},t')\, \mathscr{E}(\mathbf{k},t')\, d^3\mathbf{k}, \tag{3.9}$$

where

$$\ln \mathscr{E}(\mathbf{k},t') = -\int_0^{t'} (t'-s)\, k_i\, k_j\, H_{ij}(s)\, ds.$$

Note that when $L_{ikjl}(t-t') = \delta(t-t')\,\delta_{ik}\,\delta_{jl}$ and $\mathbf{v}_0 = 0$, (3.9) reduces to Lundgren & Pointin's (1976) equation for the Eulerian correlation tensor of fluid velocity.

The above result (3.9) can be further transformed. Denoting the symmetric and antisymmetric parts of H_{ij} by

$$H^s_{ij} = \tfrac{1}{2}(H_{ij}+H_{ji}) \quad \text{and} \quad H^a_{ij} = \tfrac{1}{2}(H_{ij}-H_{ji}), \tag{3.10}$$

respectively, with the properties

$$H^s_{ij}(s) = H^s_{ij}(-s) \quad \text{and} \quad H^a_{ij}(s) = -H^a_{ij}(-s), \tag{3.11}$$

and introducing the mean-squared random displacement of a particle

$$R_{ij}(t) \equiv \langle \rho_i(t)\,\rho_j(t)\rangle = \int_0^t dt' \int_0^t dt'' \langle v_i(t')\,v_j(t'')\rangle, \tag{3.12}$$

we obtain

$$R_{ij}(t) = \int_0^t dt' \int_{-t'}^{t-t'} H_{ij}(s)\,ds = \int_{-t}^{t} (t-|s|)\,H_{ij}(s)\,ds = 2\int_0^t (t-s)\,H^s_{ij}(s)\,ds. \tag{3.13}$$

Here we have used both the stationarity and the symmetry properties of the correlation tensor H_{ij}.

Note that the second derivative of the displacement tensor corresponds to the symmetric part only and not to the full correlation tensor, hence

$$d^2R_{ij}/dt^2 = 2H^s_{ij}(t). \tag{3.14}$$

This point, insignificant in isotropic cases, was left unnoticed by Lundgren & Pointin (1976).

Two separate equations for the symmetric and antisymmetric parts of the correlation tensor can be derived from (3.9):

$$H^s_{ij}(t) = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} \mathscr{E}(\mathbf{k},t')\,[L^s_{ikjl}(t-t')\cos(\mathbf{k}.\mathbf{v}_0 t')\,\Phi^s_{kl}(\mathbf{k},t') + L^a_{ikjl}(t-t')\sin(\mathbf{k}.\mathbf{v}_0 t')\,\Phi^a_{kl}(\mathbf{k},t')]\,d^3\mathbf{k}, \tag{3.15}$$

and

$$H^a_{ij}(t) = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} \mathscr{E}(\mathbf{k},t')\,[L^a_{ikjl}(t-t')\cos(\mathbf{k}.\mathbf{v}_0 t')\,\Phi^s_{kl}(\mathbf{k},t') + L^s_{ikjl}(t-t')\sin(\mathbf{k}\cdot\mathbf{v}_0 t')\,\Phi^a_{kl}(\mathbf{k},t')]\,d^3\mathbf{k}, \tag{3.16}$$

with

$$L^s_{ikjl} = \tfrac{1}{2}(L_{ikjl}+L_{jlik}), \quad L^a_{ikjl} = \tfrac{1}{2}(L_{ikjl}-L_{jlik}), \tag{3.17}$$

$$\Phi^s_{kl} = \tfrac{1}{2}(\Phi_{kl}+\Phi_{lk}), \quad \Phi^a_{kl} = \tfrac{1}{2}i(\Phi_{kl}-\Phi_{lk}). \tag{3.18}$$

Here the symmetry properties

$$L_{ikjl}(t) = L_{jlik}(-t), \quad \Phi_{kl}(\mathbf{k},t) = \Phi_{lk}(-\mathbf{k},-t) \tag{3.19}$$

have been used. Fortunately, (3.15) and (3.16) are not coupled, since the function $\mathscr{E}(\mathbf{k},t')$ depends only on the symmetrical part of H_{ij}. After H^s_{ij} is found by solving (3.15), H^a_{ij} can be calculated straightforwardly from (3.16). Together, (3.14) and (3.15) yield

$$\frac{1}{2}\frac{d^2R_{ij}}{dt^2} = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}\mathbf{kk}:\mathbf{R}(t')]\,[L^s_{ikjl}(t-t')\cos(\mathbf{k}.\mathbf{v}_0 t)\,\Phi^s_{kl}(\mathbf{k},t') + L^a_{ikjl}(t-t')\sin(\mathbf{k}.\mathbf{v}_0 t)\,\Phi^a_{kl}(\mathbf{k},t')]\,d^3\mathbf{k}, \tag{3.20}$$

which constitutes the principal result of this section.

The particular form corresponding to the dynamic equation (1.2) and the Green's function (2.5) is

$$\frac{d^2R_{ij}}{dt^2} = \gamma\int_{-\infty}^{\infty} e^{-\gamma|t-t'|}dt' \int_{-\infty}^{\infty} \cos(\mathbf{k}.\mathbf{v}_0 t')\,\Phi^s_{ij}(\mathbf{k},t')\exp[-\tfrac{1}{2}\mathbf{kk}:\mathbf{R}(t')]\,d^3\mathbf{k}. \tag{3.21}$$

For this particular case, the equivalent differential equation

$$\frac{d^4R_{ij}}{dt^4} - \gamma^2\frac{d^2R_{ij}}{dt^2} + 2\gamma^2\int_{-\infty}^{\infty} \cos(\mathbf{k}.\mathbf{v}_0 t)\,\Phi^s_{ij}(\mathbf{k},t)\exp[-\tfrac{1}{2}\mathbf{kk}:\mathbf{R}(t)]\,d^3\mathbf{k} = 0 \tag{3.22}$$

can be derived.

At $\gamma \to \infty$ and $\mathbf{v}_0 = 0$, i.e. when the particle follows all turbulent pulsations, this equation coincides with Lundgren & Pointin's (1976) differential equation of the Lagrangian fluid velocity correlation tensor, which should be corrected for an anisotropic case by using only the symmetric part of the spectral density $\Phi(\mathbf{k}, t)$.

4. A numerical example

For a numerical investigation we have chosen a simple isotropic system with no external forces so that there is no deterministic drift of a suspended particle. For the isotropic case the displacement tensor can be expressed using a single function:

$$R_{ij}(t) = Y(t)\,\delta_{ij}. \tag{4.1}$$

Taking the trace of (3.21) with $\mathbf{v}_0 = 0$ gives

$$\frac{d^2Y}{dt^2} = \gamma \int_{-\infty}^{\infty} e^{-\gamma|t-t'|}dt' \int_0^{\infty} \tfrac{1}{3}\Phi_{ii}(k, t') \exp\left[-\tfrac{1}{2}k^2 Y(t')\right] 4\pi k^2 dk. \tag{4.2}$$

The corresponding form of (3.22) is

$$\frac{d^4Y}{dt^4} - \gamma^2 \frac{d^2Y}{dt^2} + 2\gamma^2 \int_0^{\infty} \tfrac{1}{3}\Phi_{ii}(k, t) \exp\left[-\tfrac{1}{2}k^2 Y(t)\right] 4\pi k^2 dk = 0. \tag{4.3}$$

Further calculations can be performed using the Kraichnan model spectrum

$$\Phi_{ii}(k, t) = (2\pi)^{-1} k^{-2} E(k) \exp\left[-\tfrac{1}{2}(u_0 k_0 t)^2\right], \tag{4.4}$$

where

$$E(k) = 16(2/\pi)^{\frac{1}{2}} u_0^2 k^4 k_0^{-5} \exp\,(-2k^2/k_0^2) \tag{4.5}$$

and with the characteristic velocity u_0 defined by the normalization condition

$$\int_0^{\infty} E(k)\,dk = \tfrac{3}{2}u_0^2. \tag{4.6}$$

Substituting (4.4) and (4.5) into (4.2) and integrating over k yields

$$\frac{d^2y}{d\tau^2} = \lambda \int_{-\infty}^{\infty} e^{-\lambda|\tau-\tau'|} \frac{\exp\left[-\frac{1}{2}(\tau')^2\right]}{\left[1 + \frac{1}{4}y(\tau')\right]^{\frac{5}{2}}} d\tau', \tag{4.7}$$

where $y = k_0^2 Y$, $\tau = u_0 k_0 t$ and $\lambda = \gamma / u_0 k_0$.

The most important characteristics of the particle motion which can be calculated from the solution of (3.20) are the diffusivity

$$D_{ij} = \lim_{t \to \infty} \tfrac{1}{2} dR_{ij}/dt \tag{4.8}$$

and the kinetic energy per unit mass of the particle

$$T = \tfrac{1}{2}\langle v_i(t)\, v_i(t)\rangle = \tfrac{1}{2}H_{ii}(0) = \tfrac{1}{4}R''_{ii}(0). \tag{4.9}$$

For a particle obeying the dynamic equation (1.2), it is also possible to evaluate the correlation tensor of the random part of the relative fluid–particle velocity

$$P_{ij}(s) = \langle [v_i(t) - u_i(\mathbf{r}(t), t)]\,[v_j(t+s) - u_j(\mathbf{r}(t+s), t+s)]\rangle = \gamma^{-2}\langle v_i'(t)\, v_j'(t+s)\rangle, \tag{4.10}$$

and the rate of energy dissipation per unit mass of a particle

$$Q = \gamma \langle [v_i(t) - u_i(\mathbf{r}(t), t)]\,[v_i(t) - u_i(\mathbf{r}(t), t)]\rangle = \gamma^{-1}\langle v_i'(t)\, v_i'(t)\rangle = \gamma P_{ii}(0). \tag{4.11}$$

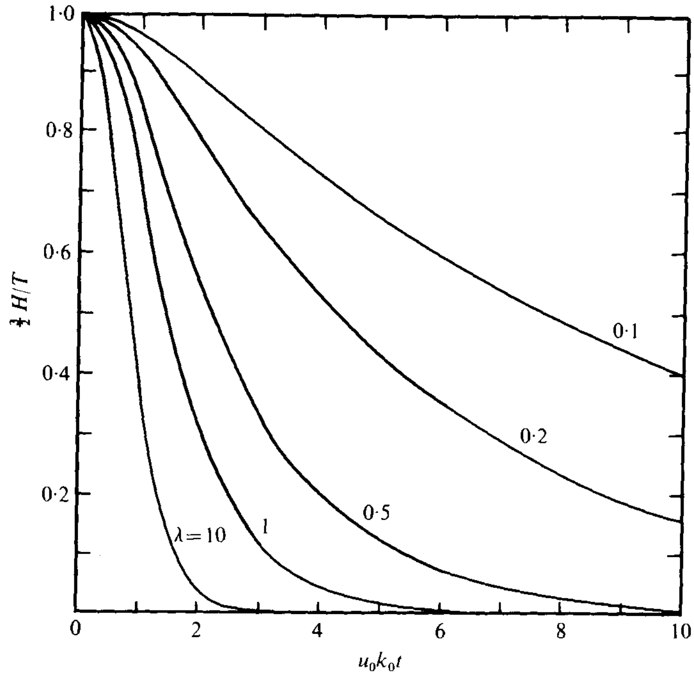

FIGURE 1. Particle velocity correlation function.

Differentiating the expression $H_{ij}(s) = \langle v_i(t)\, v_j(t+s)\rangle$ and using the stationarity of the tensors H_{ij} and P_{ij} gives

$$P_{ij}(s) = -\gamma^{-2}H''_{ij}(s), \tag{4.12}$$

hence

$$Q = -\gamma^{-1}H''_{ii}(0) = -(2\gamma)^{-1}R^{\text{iv}}_{ii}(0). \tag{4.13}$$

For the isotropic system, (4.8), (4.9), (4.12) and (4.13) can be rewritten using the non-dimensional variables defined in (4.7) in the form

$$D_{ij} = \frac{u_0}{k_0}\bar{D}\delta_{ij}, \quad \bar{D} = \lim_{\tau\to\infty}\frac{1}{2}\frac{dy}{d\tau}, \tag{4.14}$$

$$T = u_0^2\bar{T}, \quad \bar{T} = \tfrac{3}{4}y''(0), \tag{4.15}$$

$$P_{ij} = u_0^2\bar{P}\delta_{ij}, \quad \bar{P}(\tau) = -\tfrac{1}{2}\lambda^{-2}y^{\text{iv}}(\tau), \tag{4.16}$$

$$Q = u_0^3k_0\bar{Q}, \quad \bar{Q} = -\tfrac{3}{2}\lambda^{-1}y^{\text{iv}}(0). \tag{4.17}$$

A numerical solution to (4.7) subject to the initial conditions $y = 0$ and $dy/d\tau = 0$ was obtained using a CSMP simulator (Continuous System Modelling Program) and an iterative procedure. The iterations converged rapidly for all values of the parameter λ considered, provided that the integration was extended to a sufficient time interval. The physical quantities (4.14)–(4.17), as well as other functionals discussed in the next section, were then calculated straightforwardly with y^{iv} evaluated using (4.3).

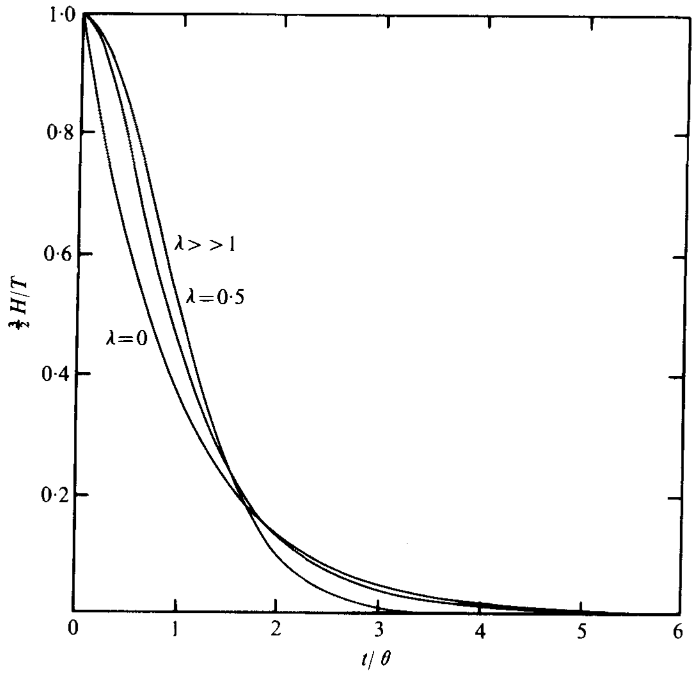

FIGURE 2. Integral-time-scale normalization of the velocity correlation function.

5. Discussion of results

The particle velocity correlation function $H(t) = \frac{1}{2}u_0^2 y''(t)$ obtained by solving (4.7) is shown in figure 1. The curves, corresponding to different values of the non-dimensional parameter $\lambda = \gamma/u_0 k_0$, are normalized by the mean particle energy per unit mass $T = \frac{3}{2}H(0)$.

All curves appear to be similarly shaped. The minor distinctions in shape become evident when the curves are superimposed by scaling the abscissa with the integral time scale

$$\theta = \tfrac{3}{2}T^{-1}\int_0^\infty H(t)\,dt$$

(figure 2). The main distinction between the curves in figure 1 is the increase in the characteristic correlation time with decreasing γ, i.e. with growing inertial effects. At the same time, the energy per unit particle mass decreases from $T = \frac{3}{2}u_0^2$ at $\lambda \to \infty$ (which corresponds to the energy of the fluid element) to $T = 0$ at $\lambda = 0$ (see figure 3). Since the particle diffusivity is proportional to the product of the integral time scale and energy, both trends compensate each other, and the diffusivity changes only slightly over the whole range of λ (figure 3).

Note that the results at $\lambda \to 0$ are exact, since in this case the particle is immobile, and $H_{ij}(s)$ can be obtained directly from (2.10) using the Eulerian correlation function of the fluid velocity $G_{ij}(t) = G_{ij}^E(0, t)$. For the particular spectrum (4.4) it gives

$$D = (\tfrac{1}{2}\pi)^{\frac{1}{2}} u_0/k_0.$$

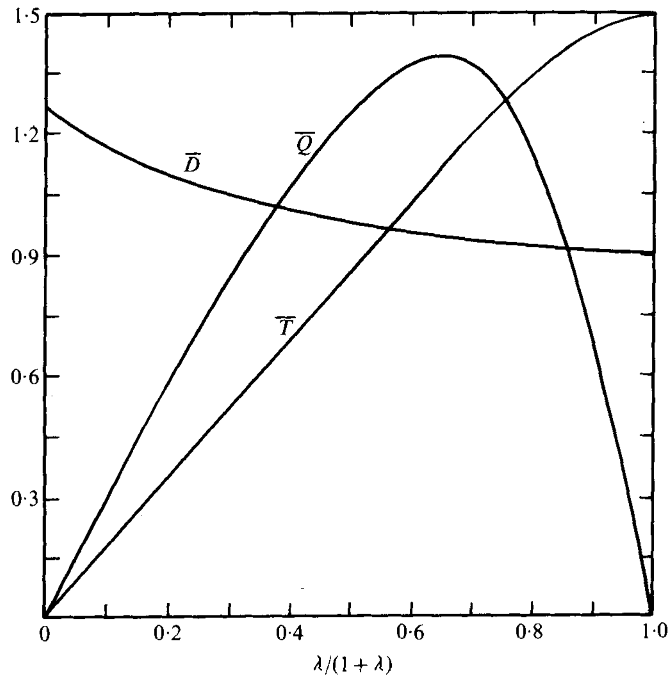

FIGURE 3. The dependence of the dissipation rate, the kinetic energy per unit particle mass and the diffusivity on the particle relative time scale λ.

Thus, it can be expected that the assumptions used for deriving (3.9) from (3.2) become more accurate with decreasing λ, and apply to the particle motion even better than to motion of fluid elements, for which they were originally suggested.†

Quite remarkable are the curves related to the fourth derivative of the mean-squared particle displacement. The correlation functions of the relative fluid–particle velocity at different λ (figure 4) do not resemble each other as closely as the particle velocity correlation functions in figure 1. All curves display a characteristic range of negative correlations. An explanation of this feature is obtained by recalling that, according to (1.2), the relative velocity correlation functions are proportional to the correlation functions of particle acceleration, and a positive acceleration of a particle at any moment makes its negative acceleration at some future moment more probable. The curves in figure 4, as well as in figure 1, show an increase in the characteristic correlation time with decreasing λ.

The most important integral characteristic of the relative fluid–particle motion is the energy dissipation rate per unit particle mass Q. As can be seen in figure 3, this quantity approaches zero at both extremes: at $\lambda \to 0$, which corresponds to a very heavy particle or to an inviscid fluid, and at $\lambda \to \infty$, when the relative fluid–particle velocity vanishes. The maximum dissipation is achieved for particles with a time

† After this paper had been submitted for publication, Dr M. Reeks kindly sent us a copy of his manuscript (1977) where the same problem was treated using an iterative procedure akin to that applied by Phythian (1975) and Levich & Pismen (1976) to turbulent self-diffusion and Brownian motion of particles in a random field, respectively. Both methods are complementary in the same manner as those of Lundgren & Pointin (1976) and Phythian (1975), and we found the numerical results to be in good agreement whenever the comparison could be made.

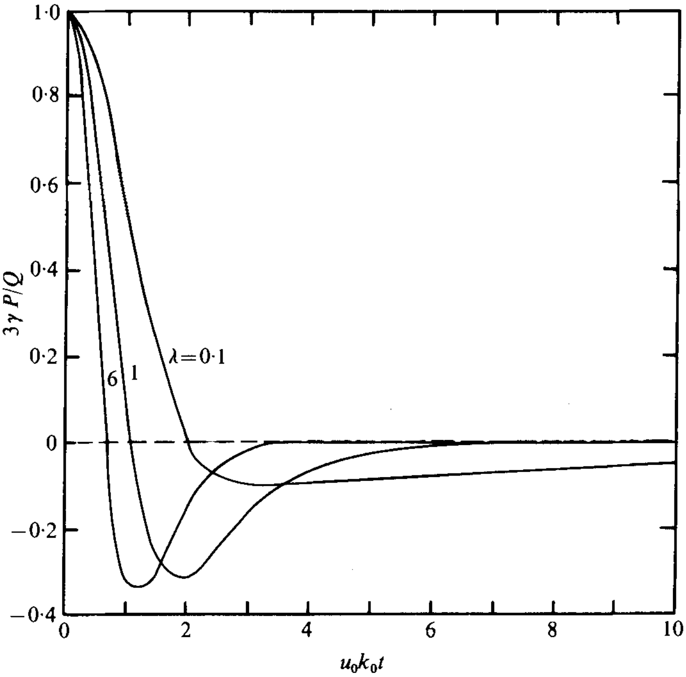

FIGURE 4. Fluid–particle relative velocity correlation function.

constant γ^{-1} of the same order of magnitude as the characteristic time $(u_0 k_0)^{-1}$ of the pulsations corresponding to the maximum of the turbulence spectrum (4.4).

Figures 5 and 6 depict an attempt to separate the contributions of different spectral components into the particle diffusivity and the rate of energy dissipation by presenting these quantities in the form

$$D = \int_{-\infty}^{\infty} \mathscr{D}(\mathbf{k})\, E(\mathbf{k})\, d^3\mathbf{k} \quad \text{and} \quad Q = \int_{-\infty}^{\infty} \mathscr{Q}(\mathbf{k})\, E(\mathbf{k})\, d^3\mathbf{k}. \qquad (5.1), (5.2)$$

Integrating (4.2) once and using (4.8) yields

$$D = \tfrac{1}{2}\int_{-\infty}^{\infty} dt \int_0^{\infty} \tfrac{1}{3}\Phi_{ii}(k,t) \exp\left[-\tfrac{1}{2}k^2 Y(t)\right] 4\pi k^2 dk. \qquad (5.3)$$

Hence, with the spectrum (4.4) and the non-dimensional variables (4.7) the function $\mathscr{D}(k)$ defined by (5.1) becomes

$$\overline{\mathscr{D}} = k_0 u_0 \mathscr{D} = \int_0^{\infty} \exp\{-\tfrac{1}{2}[\tau^2 + \chi^2 y(\tau)]\}\, d\tau, \qquad (5.4)$$

where $\chi = k/k_0$. The corresponding expression for $\mathscr{Q}$ is obtained by substituting (4.2) into (4.3) and setting $t = 0$. Hence,

$$Y^{\text{iv}}(0) = 2\gamma^2 \int_0^{\infty} [\gamma e^{-\gamma t} - \delta(t)]\, dt \int_0^{\infty} \tfrac{1}{3}\Phi_{ii}(k,t) \exp\left[-\tfrac{1}{2}k^2 Y(t)\right] 4\pi k^2 dk, \qquad (5.5)$$

and, in view of (4.4), (4.7), (4.17) and (5.2),

$$\overline{\mathscr{Q}} = \frac{\mathscr{Q}}{u_0 k_0} = 3\lambda^2 \left(1 - \int_0^{\infty} \exp\{-[\lambda\tau + \tfrac{1}{2}\tau^2 + \tfrac{1}{2}\chi^2 y(\tau)]\}\, d\tau\right). \qquad (5.6)$$

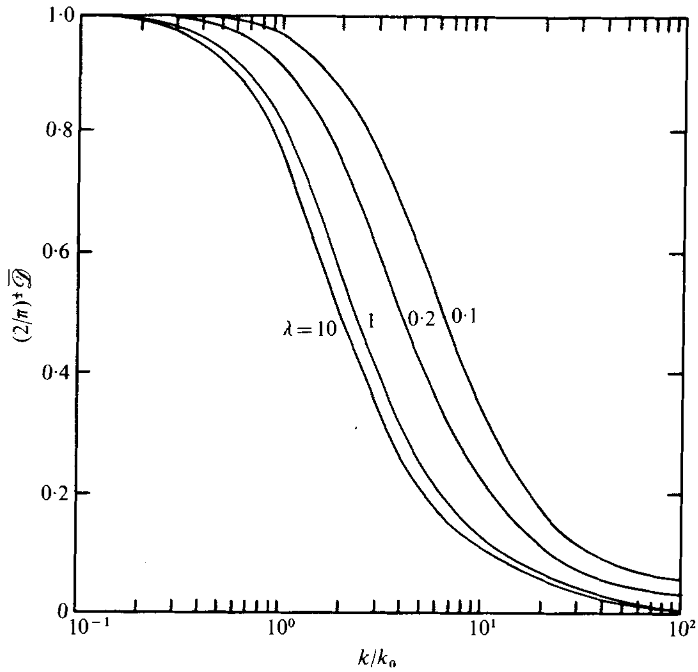

FIGURE 5. Spectral contribution to particle diffusivities.

Of course, the contributions of various spectral modes, as defined by (5.4) and (5.6), are not truly additive, since they depend on the mean-squared displacement $Y(t)$, which is influenced by the whole spectrum of turbulence. Nevertheless, the functions $\mathscr{D}$ and $\mathscr{Q}$ may help to visualize the change of contributions of different modes when the spectrum of turbulence remains fixed, but the particle time constant γ^{-1} is changing.

The curves $\bar{\mathscr{D}}(\chi;\lambda)$ in figure 5 are normalized by the maximum values of these functions at $\chi = 0$, $\bar{\mathscr{D}}(0;\lambda) = (\frac{1}{2}\pi)^{\frac{1}{2}}$. The long-wave modes, naturally, are most effective in enhancing the particle diffusivity. For relatively light particles, up to $\lambda = 1$, the curves $\bar{\mathscr{D}}(\chi;\lambda)$ remain practically unchanged. But for heavier particles, a gradual increase of the contribution of medium-range frequencies is observed.

The shape of the functions $\bar{\mathscr{Q}}(\chi;\lambda)$ (figure 6) is most remarkable. With growing λ, the contribution of any spectral mode to the energy dissipation first grows, and then starts to decline. (For modes with $k/k_0 > 5$ the maximum was not actually achieved at the highest value of λ, for which calculations were performed.) At any fixed λ, short-wave modes dissipate more effectively, but the growth of the relative dissipation rate with k is most drastic at intermediate wavelengths, with $k/k_0 = O(\lambda)$. The distinction between nearly non-dissipative long waves and intensively dissipating short-wave modes is most pronounced in the case of relatively light particles (large λ). The underlying physical fact is that any spectral mode ceases to dissipate when a particle becomes light enough to follow pulsations of the corresponding scale.

The calculations of the energy dissipation rate should be viewed with some caution, since the dissipation rate, unlike the particle displacement, is strongly affected by short-wave modes for which both independence and the Gaussian approximation become questionable. Before a rigorous analysis of the implied assumptions is per-

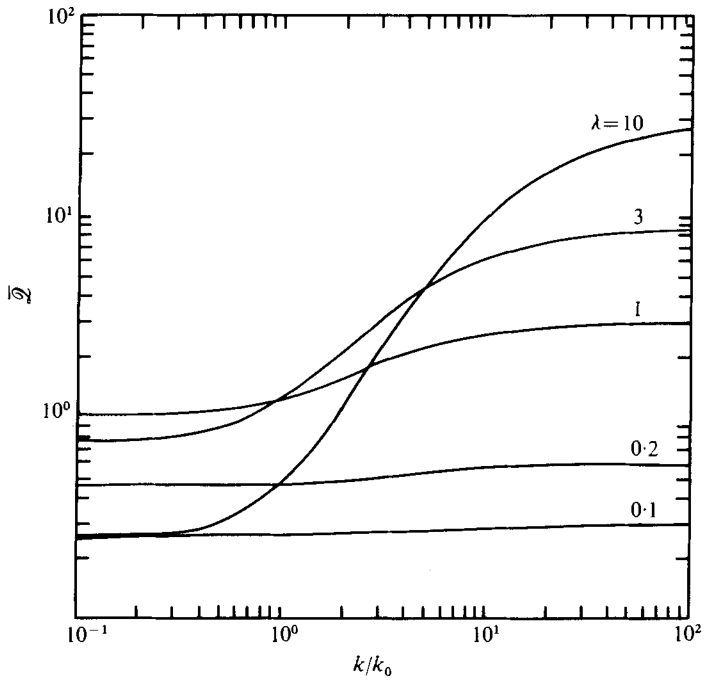

FIGURE 6. Spectral contribution to dissipation rate.

formed (e.g. along the lines of Weinstock's (1976) analysis of the problem of turbulent self-diffusion), the following tentative arguments in their favour can be considered: (*a*) the Gaussian and independence approximations are used directly only for calculating the displacement tensor, while the derivation of the dissipation rate from the displacement data is exact; (*b*) the results for the dissipation rate are exact at both extremes, $\lambda \to 0$ and $\lambda \to \infty$.

Since all calculations were performed for a system with no external force and deterministic motion, all results obtained express in pure form the influence of the inertial effects, without interference of the 'effect of crossing trajectories'. Such a system could be realized experimentally only by compensating the gravity force. Were such experiments performed, the data related to dissipative processes (e.g. rates of mass transfer between fluid and particle) would be most valuable for an examination of the theory. Their sensitivity to the character of fluid and particle motion is greater than that of measurements of the particle displacement inasmuch as they are not obscured by stronger dissipation processes connected with the deterministic motion. A more realistic anisotropic case will be considered in a following communication.

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